

Chern-Simons interactions in AdS_3 and the current conformal block

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Abstract

We compute the four point function of scalar fields in AdS_3 charged under $U(1)$ Chern-Simons fields using the bulk version of the operator state mapping. Then we show how this four point function is reproduced from a CFT_2 with a global $U(1)$ symmetry, through the contribution of the corresponding current operator in the operator product expansion, i.e. through the conformal block of the current operator. We work in a "probe approximation" where the gravitational interactions are ignored, which corresponds to leaving out the energy momentum tensor from the operator product expansion.

1 Introduction

Chern-Simons (CS) theory provides a beautiful example of a holographic duality. Chern-Simons theory on a manifold with a boundary can be directly related to a 1+1 dimensional CFT on the boundary, the Wess-Zumino-Witten (WZW) model [1, 2]. The gauge symmetry of the bulk Chern-Simons theory is related to the global symmetry of the WZW model, much like in the AdS/CFT duality.

In the context of the $\text{AdS}_3/\text{CFT}_2$ duality, Chern-Simons gauge fields naturally arise in the bulk as the low energy limit of string theory [3, 4]. As is well known, even gravity in AdS_3 (at least classically) can be written as a Chern-Simons theory [5, 6].

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As discussed recently in [7], much of the structure of the original CS/WZW duality generalize to the examples arising in the AdS/CFT duality, although important differences also appear [4].

One particular new feature that arises in the AdS/CFT examples is that generically one has dynamical matter in the bulk charged under the Chern-Simons fields. The purpose of this note is to study the correlation functions of CFT operators dual to scalar fields in the bulk charged under the CS fields. A nice feature of the Chern-Simons interaction is that the calculation of the three and four point functions is considerably simpler (one might even say trivial), than in higher dimensional examples (see e.g. [8]). Thus, they provide simple examples of higher point correlation functions that are easily calculable in AdS.

The original CS/WZW duality is particularly nice as the "dictionary" between the bulk and the boundary is explicit. Thus, bulk results can be straightforwardly understood from the boundary perspective as well. This is unlike in general examples of AdS/CFT where even simple bulk physics, such as locality for example, can be complicated from the boundary theory perspective (even though considerable recent progress has been made towards understanding bulk locality see e.g. [9, 10]). We find that this simplicity at least partially continues to the case with dynamical charged matter, as we can give an interpretation of the Chern-Simons interaction in the four point function very simply within the CFT. The CS interaction can be reproduced using the operator product expansion (OPE) as follows. The CFT has a current operator dual to the CS gauge fields. The contribution of the current operator and its conformal descendants to the OPE can be resummed to give rise to what is called the conformal block of the current operator. The current conformal block precisely reproduces the effect of the bulk CS interaction from the CFT. This is unlike in more general examples of the AdS/CFT duality [11], where the conformal block of a given operator is known not to be enough to reproduce the exchange interaction of the corresponding dual bulk field.

Throughout this note we will work in a "probe approximation" where bulk gravity is ignored, and also we treat the Chern-Simons interaction perturbatively in powers of $1/k$, where k is the Chern-Simons level. Thus, we are assuming $c \gg k \gg 1$, where c is the central charge of the CFT. Also we only consider $U(1)$ Chern-Simons theory, while generalization to non-Abelian groups seems straightforward.

The current manuscript is organized as follows. In section 2 we provide a short introduction to classical Chern-Simons theory in AdS_3 , and compute two point functions of the corresponding dual current operators. In section 3 we discuss the operator

state mapping in the context of AdS/CFT, which will be used in the rest of the paper. In section 4 we compute a three point function of charged scalars and the current as a warm up. In section 5 we compute the four point function of charged scalar operators. In section 6, we introduce the OPE formalism in the CFT and show that the current operator conformal block reproduces the bulk four point function. In section 7 we perform a simple check of the results by computing the interaction energy between charged bulk particles and compare the result to the CFT anomalous dimension. In section 8 we discuss the relation of the CS interaction to more complicated bulk interactions.

2 Review of Chern-Simons Holography

The Euclidean action of a single Chern-Simons field is

$$S_E = -\frac{k}{8\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + S_{bdy}. \quad (1)$$

Here we use complex coordinates $z = x + i\tau$, $\bar{z} = x - i\tau$ in terms of which the metric of AdS_3 is

$$ds^2 = \frac{1}{u^2} (du^2 + dz d\bar{z}). \quad (2)$$

From this point on, we will use the gauge $A_u = 0$. The non-vanishing gauge field components are thus A_z and $A_{\bar{z}}$. The equations of motion $F_{\mu\nu} = 0$, in this gauge imply that A_z and $A_{\bar{z}}$ are both independent of u and that

$$F_{z\bar{z}} = 0. \quad (3)$$

This can be solved by setting ¹

$$A_z = \partial_z \Lambda(z, \bar{z}), \quad A_{\bar{z}} = \partial_{\bar{z}} \Lambda(z, \bar{z}). \quad (4)$$

Now we have exhausted all the equations of motion. Next we should specify boundary conditions. Specifying both A_z and $A_{\bar{z}}$ as boundary conditions is overconstraining as for generic A_z and $A_{\bar{z}}$ no $\Lambda(z, \bar{z})$ satisfying (4) exists. Thus, we are lead to fix one combination of the two as a boundary condition. We will choose to fix $A_{\bar{z}}$. In the following we will see that this is indeed a consistent boundary condition with a simple

¹Topologically Euclidean AdS_3 is the product of a disk and a real line. Thus, all closed forms in AdS_3 are also exact.

physical meaning in the boundary CFT.

For the boundary condition to be consistent with the variational principle, we need to ensure that the boundary terms in the variation of the action vanish. The variation of the Chern-Simons action is

$$\delta S_E = \frac{k}{8\pi} \int d^2z \left(A_{\bar{z}} \delta A_z - A_z \delta A_{\bar{z}} \right) + \delta S_{bdy}. \quad (5)$$

Gauge field configurations within the space of allowed variations (respecting our boundary conditions) satisfy $\delta A_{\bar{z}} = 0$, leading to the variation

$$\delta S_E = \frac{k}{8\pi} \int d^2z \delta A_z A_{\bar{z}} + \delta S_{bdy}. \quad (6)$$

If we choose the following covariant boundary term

$$S_{bdy} = -\frac{k}{16\pi} \int d^2z \sqrt{|\gamma|} A^2 = -\frac{k}{8\pi} \int d^2z A_z A_{\bar{z}}, \quad (7)$$

the variation of the action vanishes identically $\delta S_E = 0$. If we had chosen the boundary condition $\delta A_z = 0$, the boundary term would have the opposite sign.

Within the holographic dictionary, $A_{\bar{z}}$ is identified as a background gauge field coupled to a left moving current j_z . According to the usual holographic dictionary

$$\langle j_z \rangle = \frac{\delta S}{\delta A_{\bar{z}}} = -\frac{k}{4\pi} A_z|_{u=0}. \quad (8)$$

The current two point function can be obtained from the current one point function in the presence of a delta function source for the gauge field. To see this we write the current expectation value with the delta function source as

$$\langle j_z(z') \rangle_{A_{\bar{z}}=\delta^{(2)}(z)} = \frac{\langle j_z(z') e^{-j_z(0)} \rangle}{\langle e^{-j_z(0)} \rangle}, \quad (9)$$

where the latter expectation values are taken in the CFT ground state. Expanding the exponential and using large- N factorization of the correlation functions (which we are assuming here) gives

$$\langle j_z(z') \rangle_{A_{\bar{z}}=\delta^{(2)}(z)} = -\langle j_z(z') j_z(0) \rangle. \quad (10)$$

Thus, to obtain the holographic two point function, we need to find a solution to the Chern-Simons equations of motion that satisfies $A_{\bar{z}} = \delta^{(2)}(z)$. The corresponding

solution is of the form (4) with

$$\Lambda(z, \bar{z}) = \frac{1}{2\pi z}, \quad (11)$$

due to the identity $\partial_{\bar{z}}(1/(2\pi z)) = \delta^{(2)}(z, \bar{z})$. The solution (11) is not unique in the sense that naively we could add an arbitrary function of z to it, while satisfying the boundary conditions and the equation of motion. But requiring that the only singularity in $A_{\bar{z}}$ is a delta function at $z = 0$, fixes the function of z to be at most a constant. Thus, we obtain the current two point correlator

$$\langle j_z(z) j_z(0) \rangle = \frac{k}{4\pi} \partial_z \Lambda = -\frac{k}{8\pi^2} \frac{1}{z^2}. \quad (12)$$

This is not quite the theory we want to consider in the following, since the current j_z is not really conserved since (8), together with the equation of motion (3), implies that in the presence of a background gauge field one has

$$\langle \partial_{\bar{z}} j_z \rangle = -\frac{k}{4\pi} \partial_z A_{\bar{z}}. \quad (13)$$

This is the chiral anomaly discussed in detail in [7].

We would like to consider a case when the current in the dual field theory is fully conserved. To achieve this, we should introduce another component $j_{\bar{z}}$ for the current to obtain a parity invariant theory and to get rid of the anomaly. For this purpose we can introduce another gauge field \bar{A} interpreted as being dual to $j_{\bar{z}}$. The full action we choose is

$$S_E = -\frac{k}{8\pi} \int d^3x \epsilon^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda - \bar{A}_\mu \partial_\nu \bar{A}_\lambda) - \frac{k}{8\pi} \int d^2z (A_z A_{\bar{z}} + \bar{A}_z \bar{A}_{\bar{z}} - 2A_{\bar{z}} \bar{A}_z). \quad (14)$$

For \bar{A} to be dual to a right moving current, we fix \bar{A}_z at the boundary. The \bar{A} boundary term in (14) is chosen again to obtain a well defined variational principle. The coefficient of the last boundary term is chosen to make the current conserved, as can be easily checked.

Again the current correlator is given by the current one point function in the presence of a delta function source. The current correlation functions are now given

by

$$\begin{aligned} G_{zz} &= -\frac{k}{8\pi^2} \frac{1}{z^2}, & G_{\bar{z}\bar{z}} &= -\frac{k}{8\pi^2} \frac{1}{\bar{z}^2}. \\ G_{z\bar{z}} &= G_{\bar{z}z} = -\frac{k}{8\pi^2} \partial_z \left(\frac{1}{\bar{z}} \right). \end{aligned} \quad (15)$$

Above we defined $G_{ab} = \langle j_a(z) j_b(0) \rangle$. The $G_{z\bar{z}}$ component appeared from the last boundary term in (14).

Neglecting the contact term, the two point function can be written compactly in terms of cartesian coordinates on R^2 as

$$G_{ab} = \frac{C_V}{x^2} \left(\delta_{ab} - 2 \frac{x_a x_b}{x^2} \right), \quad C_V = \frac{k}{4\pi^2}. \quad (16)$$

3 Operator state mapping in AdS/CFT

A CFT on the complex plane, with a complex coordinate z , can be mapped to the cylinder using the standard conformal map

$$w = \log z, \quad w = \tau + i\phi \quad (17)$$

Now a dilatation on the plane $z \rightarrow e^\lambda z$ takes the form of a time translation $\tau \rightarrow \tau + \lambda$ on the cylinder. Thus, the dilatation operator becomes the Hamiltonian on the cylinder. Under the conformal transformation (17), correlation functions of primary operators transform as

$$\langle \mathcal{O}_1(w_1, \bar{w}_1) \dots \mathcal{O}_n(w_n, \bar{w}_n) \rangle = \prod_{j=1}^n \left(\frac{\partial z_j}{\partial w_j} \right)^{h_j} \left(\frac{\partial \bar{z}_j}{\partial \bar{w}_j} \right)^{\bar{h}_j} \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_n(z_n, \bar{z}_n) \rangle, \quad (18)$$

where (h_j, \bar{h}_j) are the conformal dimensions of the operators \mathcal{O}_j . The operators we will be mainly considering are scalar operators for which $(h_j, \bar{h}_j) = (\Delta/2, \Delta/2)$, and the left moving current j_z with $(h_j, \bar{h}_j) = (1, 0)$ and the right moving current $j_{\bar{z}}$ with $(h_j, \bar{h}_j) = (0, 1)$. The CFT vacuum state corresponds to a Euclidean path integral with no operators located at $z = 0$. Whenever an operator approaches $z = 0$ it can be absorbed into the definition of the initial state $\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle$. This is a dilatation eigenstate as can be seen from the conformal symmetry algebra, and thus, an energy eigenstate on the cylinder. Similarly, as an operator approaches $z \rightarrow \infty$, it can be

absorbed into the final state [12, 13]

$$\lim_{z \rightarrow \infty} \bar{z}^{2h} z^{2\bar{h}} \langle 0 | \mathcal{O}(\bar{z}, z) = \langle \mathcal{O} |, \quad (19)$$

which is an energy eigenstate on the cylinder for the same reason.

In the context of AdS/CFT, a CFT on a plane is dual to AdS in the Poincare coordinates

$$ds^2 = \frac{1}{u^2} (du^2 + dz d\bar{z}), \quad (20)$$

while a CFT on a cylinder is dual to AdS in global coordinates

$$ds^2 = \frac{1}{\cos^2 \theta} (d\tau^2 + d\theta^2 + \sin^2 \theta d\phi^2). \quad (21)$$

These two metrics are related by a coordinate transformation

$$\theta = \arctan \left(\frac{|z|}{u} \right), \quad w = \frac{1}{2} \log \left[z^2 \left(1 + \frac{u^2}{|z|^2} \right) \right], \quad (22)$$

where $w = \tau + i\phi$. Indeed at the boundary $u \rightarrow 0$, the coordinate transformation reduces to the conformal map (17) from the plane to the cylinder.

To see how the operator state mapping works, consider a scalar operator inserted at $z = 0$ in the CFT. In the leading order in the large- N limit the operator insertion is identical to turning on a delta function source for the corresponding operator at $z = 0$. In the bulk (Poincare patch), the solution approaching a delta function at the boundary is the bulk to boundary propagator

$$\varphi_0(u, z, \bar{z}) = C_\Delta \frac{u^\Delta}{(u^2 + z\bar{z})^\Delta}. \quad (23)$$

Inverting the coordinate transformation (22)

$$|z| = \sin \theta e^\tau, \quad u = \cos \theta e^\tau, \quad (24)$$

the bulk field configuration becomes²

$$\varphi_0(\tau, \phi, \theta) = \frac{1}{\sqrt{2\pi}} e^{-\Delta\tau} (\cos \theta)^\Delta. \quad (25)$$

When continued to real time, this is the wavefunction of the lowest energy eigenstate of the bulk scalar field in global AdS (see e.g. [14]). When canonically quantizing the

²We have chosen the overall normalization so that φ_0 has a unit Klein-Gordon norm.

bulk scalar, one uses the mode expansion

$$\varphi(x) = \sum_{n,l} (f_{n,l}(x)a_{n,l} + f_{n,l}^*(x)a_{n,l}^\dagger). \quad (26)$$

The mode functions $f_{n,l}$ are labeled by two quantum numbers, the radial mode number n and the angular momentum l . The lowest energy state corresponds to the one we obtained through the operator state mapping $f_{0,0} = \varphi_0$. We will denote the corresponding bulk quantum state as $a_{0,0}^\dagger|0\rangle = |1\rangle$. The insertion of a primary operator corresponds to preparing the corresponding particle in its lowest energy eigenstate. Thus, the holographic dictionary between the bulk and the boundary states is

$$|\mathcal{O}\rangle \leftrightarrow |1\rangle. \quad (27)$$

The excited states created by $a_{n,l}^\dagger$ correspond to the (global) conformal descendant operators of the form

$$(\partial_\mu \partial^\mu)^n \partial_{\mu_1} \dots \partial_{\mu_l} \mathcal{O}(0), \quad (28)$$

We refer the interested reader to [14] for more details. Here we will only need the ground state wavefunction. In 1+1 CFT in addition to (28) there are also higher Virasoro descendants of the primary operators. In the bulk, these correspond to adding boundary graviton excitations. We will work in the "probe approximation", where these are ignored.

4 A warm up three point function

Next we couple a charged scalar field to the Chern-Simons theory. The action for the scalar field is chosen as

$$S_{scalar} = \int d^3x \sqrt{g} (D\varphi^\dagger D\varphi + m^2 \varphi^\dagger \varphi), \quad (29)$$

where $D_\mu \varphi = \partial_\mu \varphi - i\frac{e}{2}(A_\mu + \bar{A}_\mu)\varphi$. In the following we will apply the operator state mapping to the three point function

$$G_3 = \langle \mathcal{O}^\dagger(z_1, \bar{z}_1) j_z(z_2, \bar{z}_2) \mathcal{O}(z_3, \bar{z}_3) \rangle, \quad (30)$$

where \mathcal{O} is the operator dual to φ . The form of all three points functions in a CFT are fixed by conformal symmetry up to an overall constant. In the current case

$$G_3 = \frac{C_{\mathcal{O}^\dagger j \mathcal{O}}}{2} \frac{1}{|z_{13}|^{2\Delta}} \left(\frac{1}{z_{23}} - \frac{1}{z_{21}} \right). \quad (31)$$

Above we introduced the notation $z_{ij} = z_i - z_j$. Taking a limit where the scalar operators approach $z_3 \rightarrow \infty$ and $z_1 \rightarrow 0$ and using (18) and (19), we obtain the OPE coefficient in terms of a one point function on the cylinder

$$C_{\mathcal{O}^\dagger j \mathcal{O}} = 2 \langle \mathcal{O} | j_w | \mathcal{O} \rangle. \quad (32)$$

Thus, we are lead to calculate the expectation value of the current operator j_w in the lowest energy one particle state $|\mathcal{O}\rangle$. This state in the bulk is simply

$$|1\rangle = a_{0,0}^\dagger |0\rangle, \quad (33)$$

with the wavefunction (25). This state sources the gauge fields A_μ and \bar{A}_μ as dictated by the Chern-Simons equation of motion

$$\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \bar{A}_\lambda = \sqrt{-g} J^\mu, \quad (34)$$

$$\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = -\sqrt{-g} J^\mu, \quad (35)$$

where J^μ is the current generated by the ground state wavefunction

$$J^\mu = -ig^{\mu\nu} (\varphi_0^* \partial_\nu \varphi_0 - \varphi_0 \partial_\nu \varphi_0^*), \quad (36)$$

explicitly given by

$$J^t = \frac{e\Delta}{2\pi} (\cos \theta)^{2\Delta+2}, \quad J^\theta = J^\phi = 0. \quad (37)$$

In (34) and (35), we have neglected the $(A + \bar{A})^2 \phi^\dagger \phi$ term as it gives contributions of the order $1/k^2$. In the gauge $A_\theta = 0$, the unique solution³ to equation (34), and the corresponding equation for A_μ , respecting "normalizable" boundary conditions is

$$A = \frac{e}{k} dt + \frac{e}{k} (1 - (\cos \theta)^{2\Delta}) d\phi, \quad (38)$$

$$\bar{A} = \frac{e}{k} dt - \frac{e}{k} (1 - (\cos \theta)^{2\Delta}) d\phi. \quad (39)$$

³Up to gauge transformations that vanish at the boundary and at the past and the future infinity.

In particular, as we approach the boundary, the gauge fields asymptote to

$$A = \frac{e}{k}(dt + d\phi), \quad \bar{A} = \frac{e}{k}(dt - d\phi). \quad (40)$$

Continuing to Euclidean time the boundary values of the fields become

$$A_w = -i\frac{e}{k} = \bar{A}_{\bar{w}}. \quad (41)$$

Using the holographic dictionary we obtain

$$\langle \mathcal{O}|j_w|\mathcal{O} \rangle = -\frac{k}{4\pi}A_w = i\frac{e}{4\pi}, \quad \langle \mathcal{O}|j_{\bar{w}}|\mathcal{O} \rangle = -\frac{k}{4\pi}\bar{A}_{\bar{w}} = i\frac{e}{4\pi}. \quad (42)$$

Comparison to (32) gives us the OPE coefficient

$$C_{\mathcal{O}^\dagger j \mathcal{O}} = i\frac{e}{2\pi}, \quad (43)$$

which is indeed the correct result as it is fixed by a current conservation Ward identity as discussed in Appendix B. The bulk version of this statement is that the asymptotic values of the gauge fields (40) are fixed by the total charge in the corresponding bulk state.

5 Four point function of charged operators

The bulk version of the operator state mapping illustrated in the previous section becomes particularly powerful in the case of 4-point functions. The operator state mapping has been earlier used in [15] to calculate four point functions in higher spin gravity in AdS_3 . The following calculation is very similar in spirit to that of [15].

We will consider a system where in the bulk there are two scalar fields φ_1 and φ_2 that only interact through the Chern-Simons fields. Other interactions would give rise to additive corrections to our result, in the first order in bulk perturbation theory. We will consider the 4-point function of the corresponding dual scalar operators

$$G_4 = \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_1^\dagger(z_2, \bar{z}_2) \mathcal{O}_2^\dagger(z_3, \bar{z}_3) \mathcal{O}_2(z_4, \bar{z}_4) \rangle. \quad (44)$$

This correlation function is neither fixed by conformal invariance nor by Ward identities. Conformal invariance guarantees that the 4-point function can be written in

the (standard) form

$$G_4 = \frac{1}{|z_{12}|^{2\Delta_1} |z_{23}|^{2\Delta_2}} g(u, v), \quad (45)$$

where $z_{ij} = z_i - z_j$, and u and v are the (global) conformal invariant cross ratios

$$u = \left| \frac{z_{12} z_{34}}{z_{13} z_{24}} \right|^2, \quad v = \left| \frac{z_{14} z_{23}}{z_{13} z_{24}} \right|^2. \quad (46)$$

Thus, conformal symmetry alone does not constrain the form of the function $g(u, v)$. Next we would like to perform the operator state mapping and take $z_3 \rightarrow \infty$ and $z_4 \rightarrow 0$ to absorb the \mathcal{O}_2 and \mathcal{O}_2^\dagger operators into the initial and final states. Using the transformation law (17) of the correlator together with (19) leads to

$$g(u, v) = \frac{\langle \mathcal{O}_2 | \mathcal{O}_1(\tau_1, \phi_1) \mathcal{O}_1^\dagger(\tau_2, \phi_2) | \mathcal{O}_2 \rangle}{\langle 0 | \mathcal{O}_1(\tau_1, \phi_1) \mathcal{O}_1^\dagger(\tau_2, \phi_2) | 0 \rangle}, \quad (47)$$

with

$$v = e^{2(\tau_1 - \tau_2)}, \quad u = 1 - 2e^{\tau_1 - \tau_2} \cos(\phi_1 - \phi_2) + e^{2(\tau_1 - \tau_2)}. \quad (48)$$

Thus, we have reduced the full 4-point function into a two point function in the cylinder, in the state $|\mathcal{O}_2\rangle$.

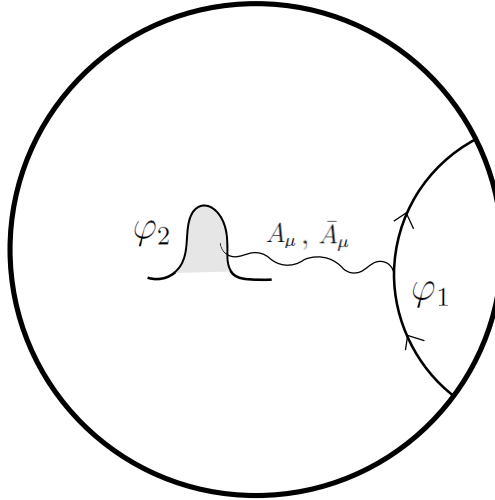


Figure 1: The operator state mapping relates the four point function into a two point function of φ_1 in the background of the φ_2 ground state. The grey blob represents the φ_2 ground state wavefunction.

For practical purposes we will again work in real time. In the bulk the procedure of the operator state mapping means that we prepare φ_2 into its ground state and

compute the two point function of φ_1 in this background. This is visualized in Figure 1. When the Chern-Simons interaction is the only interaction between the fields, we are lead to calculate the φ_1 two point function in the background Chern-Simons fields given by (38) and (39).⁴ The final simplification happens due to the fact that φ_1 couples only to the combination $A + \bar{A}$. This combination is simply a constant

$$A + \bar{A} = \frac{2e_2}{k}dt = 2\mu dt. \quad (49)$$

This indeed follows directly from adding together the equations (34) and (35), leading to an unsourced Chern-Simons equation $d(A + \bar{A}) = 0$.

The overall effect of the state $|\mathcal{O}_2\rangle$ is thus the following. It introduces a constant background chemical potential $\mu = e_2/k$ for the scalar field φ_1 . Now the beauty of this is that we do not even have to calculate the φ_1 two point function to find the full 4-point function! To obtain the two point function we should find the complete set of solutions to the equation⁵

$$\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\varphi_1) + g^{tt}(\partial_t - ie_1\mu)^2\varphi_1 - m_1^2\varphi_1 = 0. \quad (50)$$

Let us define a new field φ' through $\varphi_1 = e^{ie_1\mu t}\varphi'$. Plugging this into (50) leads to a free Klein-Gordon equation for φ' without a chemical potential. So the effect of the Chern-Simons interaction is to shift the energies of all states of the φ_1 particle in the bulk by a constant $e_1\mu$. Thus, the bulk two point function of φ_1 is given by

$$e^{ie_1\mu(t_1-t_2)}\langle 0|\varphi'(x_1)\varphi'(x_2)^\dagger|0\rangle. \quad (51)$$

The boundary two point function can be obtained as the boundary limit of the bulk two point function. Continuing back to Euclidean time $\tau = it$ and plugging this to the formula for the 4-point function (47) we see that everything else cancels out except

$$g(u, v) = e^{e_1\mu(\tau_1-\tau_2)} = v^{e_1\mu/2}. \quad (52)$$

Since we are working in the first order in an expansion in $1/k$, we are only allowed to keep the first order in an expansion in μ

$$g(u, v) = 1 + \frac{e_1e_2}{2k}\log v + O(1/k^2). \quad (53)$$

⁴One should note that this prescription actually leaves out disconnected self-energy contributions to the four point function. We will ignore these contribution everywhere in this manuscript.

⁵Or alternatively the bulk to boundary propagator.

Thus, to first order in $1/k$ the boundary four point function is given by

$$G_4 = \frac{1}{|z_{12}|^{2\Delta_1}|z_{23}|^{2\Delta_2}} \left(1 + \frac{e_1 e_2}{2k} \log v + O(1/k^2) \right). \quad (54)$$

6 The CFT side of the story

The CFTs dual to weakly coupled bulk theories exhibit factorization of correlation functions of appropriately defined operators. In the case of matrix field theories, the operators are appropriately normalized single trace operators. Here we will assume a CFT that has at least two (single trace) operators \mathcal{O}_1 and \mathcal{O}_2 , with corresponding scaling dimensions Δ_1 and Δ_2 .

It is useful to organize the boundary theory four point function in terms of the operator product expansion. Operator product expansion is a general property of local quantum field theories. Consider the correlation function

$$\langle \mathcal{O}_i(0) \mathcal{O}_j(x) \dots \rangle, \quad (55)$$

where the dots denote possible other operators. Operator product expansion states that in the limit $x \rightarrow 0$, the above correlation function can be expanded in a series of local operators $\mathcal{O}_n(x)$ in the form

$$\langle \mathcal{O}_i(0) \mathcal{O}_j(x) \dots \rangle = \sum_n f_{ijn}(x) \langle \mathcal{O}_n(x) \dots \rangle, \quad (56)$$

Conformal symmetry fixes the x dependence of $f_{ijn}(x)$ into a power law. Furthermore, in a conformal field theory the operators $\mathcal{O}_n(x)$ can be classified into conformal primary operators and their descendants (obtained by acting on the primaries with Virasoro generators). Thus, the operator product expansion in a CFT can be organized in terms of a sum over conformal primaries only, with more complicated coefficient functions f_{ijn} that include derivative operators.

The OPE can be applied twice to the four point function G_4 as $|z_{12}| \rightarrow 0$ and $|z_{34}| \rightarrow 0$. This reduces the four point function into a double sum over primary operators

$$G_4 = \sum_{n,m} D_{11^\dagger n}(x_{12}, \partial_1) D_{2^\dagger 2m}(x_{34}, \partial_3) \langle \mathcal{O}_n(x_1) \mathcal{O}_m(x_3) \rangle. \quad (57)$$

The derivative operators D_{ijn} are fixed by conformal symmetry up to an overall normalization, which is the OPE coefficient C_{ijn} . The double sum can be reduced

into a single sum due to the orthogonality of the primary two point functions. The contribution of each primary operator within the sum is called a conformal block of the corresponding operator.

Let us apply this to our setting. We are assuming that we have a theory that has the scalar operators and in addition at least the unit operator 1 and the current operator j^μ . Here we will show that the presence of these operators is enough to reproduce the physics of scalar fields coupled to Chern-Simons theory in the bulk. The following discussion still applies to more general CFT's, in which case the other operators give additional contributions to the 4 point function.

The OPE in this case reads

$$\mathcal{O}_i^\dagger(z_1)\mathcal{O}_i(z_2) = \frac{1}{|z_{12}|^{2\Delta_i}} \left(1 + \sum_{n=0}^{\infty} c_n^i z_{12}^{n+1} \partial_{z_2}^n j_z(z_2) + h.c. + \dots \right), \quad (58)$$

where the first term corresponds to the unit operator and the sum corresponds to the current operator and its descendants. The OPE coefficients are worked out in Appendix C and are given by

$$c_n^i = \frac{C_{\mathcal{O}_i^\dagger j \mathcal{O}_i}}{(n+1)! C_V} \quad (59)$$

Above we have only included the global conformal descendants of the current operator, while in a 1+1 CFT, as the conformal symmetry algebra gets enhanced to the Virasoro algebra, there are also Virasoro descendants of the current operator present. The reason for neglecting these operators from the OPE is that their contributions are suppressed at large central charge. A more detailed discussion is included in Appendix D.

Using the OPE (58) twice within the 4 point function (44) gives

$$G_4 = \frac{1}{|z_{12}|^{2\Delta_1} |z_{34}|^{2\Delta_2}} \left(1 + \sum_{n,m} (c_n^1)^* c_m^2 z_{12}^{n+1} z_{34}^{m+1} \partial_{z_2}^n \partial_{z_4}^m \langle j_z(z_2) j_z(z_4) \rangle + c.c. \right). \quad (60)$$

Thus, we should perform the double sum

$$S = \sum_{n,m} \frac{z_{12}^{n+1} z_{34}^{m+1}}{(n+1)!(m+1)!} \partial_{z_2}^n \partial_{z_4}^m \frac{1}{z^2} = \left(\sum_{n=0}^{\infty} \frac{z_{12}^{n+1}}{(n+1)!} \partial_{z_2}^n \right) \left(\sum_{m=0}^{\infty} \frac{z_{34}^{m+1}}{(m+1)!} \partial_{z_4}^m \frac{1}{z_{24}^2} \right) \quad (61)$$

Using the identity $\partial^n(1/z^2) = (-1)^n(n+1)!z^{-n-2}$ we can perform the sum over m

first to give

$$S = \sum_{n=0}^{\infty} \frac{z_{12}^{n+1}}{(n+1)!} \partial_{z_2}^n \left(\frac{1}{z_{23}} - \frac{1}{z_{24}} \right) \quad (62)$$

Then, using $\partial_z^n (1/z) = (-1)^n n! z^{-n-1}$ we get

$$S = \log \left(\frac{z_{13} z_{24}}{z_{23} z_{14}} \right). \quad (63)$$

Thus, the current operator and its descendants give the following contribution to the four point function

$$G_4 = \frac{1}{|z_{12}|^{2\Delta_1} |z_{34}|^{2\Delta_2}} \left(1 + \frac{C_{\mathcal{O}_1^\dagger j \mathcal{O}_1}^* C_{\mathcal{O}_2^\dagger j \mathcal{O}_2}}{2C_V} \log v + \dots \right), \quad (64)$$

where the dots denote possible other operators in the CFT. The function $\log v$ is the conformal block of the current operator [16]. Plugging in the values of the OPE coefficients and of C_V , gives

$$G_4 = \frac{1}{|z_{12}|^{2\Delta_1} |z_{34}|^{2\Delta_2}} \left(1 + \frac{e_1 e_2}{2k} \log v \right). \quad (65)$$

This indeed is exactly what we found from bulk Chern-Simons theory.

7 A simple check of the results

As a simple check of the results for the 4-point functions we show how it reproduces the binding energies of charged particles in the bulk. From the perspective of the CFT on a plane, the binding energy of two bulk particles φ_1 and φ_2 is given by the difference of the scaling dimension of the composite operator $\mathcal{O}_1 \mathcal{O}_2$ from the free result $\Delta_1 + \Delta_2$. To obtain this scaling dimension we consider the two point function of composite operators

$$G_2 = \langle (\mathcal{O}_1 \mathcal{O}_2)(z, \bar{z}) (\mathcal{O}_1 \mathcal{O}_2)^\dagger(z', \bar{z}') \rangle. \quad (66)$$

This can be obtained from G_4 as a limit⁶ where $|z_{14}| = |z_{23}| = \epsilon \rightarrow 0$, while $|z_{12}| \approx |z - z'|$. In this limit, after using $v \approx \epsilon^4/|z - z'|^4$, the four point function becomes

$$G_4 = \frac{1}{|z - z'|^{2\Delta_1+2\Delta_2}} \left(1 - \frac{e_1 e_2}{k} \log \frac{|z - z'|^2}{\epsilon^2} + \dots \right), \quad (67)$$

The logarithmic term can be recognized as a first term in the series of the composite operator two point function

$$\frac{1}{|z - z'|^{2(\Delta_1+\Delta_2+\gamma)}}, \quad (68)$$

in powers of the anomalous dimension γ . Thus, we can identify

$$\gamma = \frac{e_1 e_2}{k}. \quad (69)$$

Next we consider the binding energy from the bulk point of view. We start by considering classical charged particles in the bulk. A classical particle with charge e at the center of AdS_3 gives rise to a current

$$\sqrt{-g} J^\mu = e \frac{dx^\mu}{dt} \delta^{(2)}(x). \quad (70)$$

The Chern-Simons equations of motion then give

$$F_{\theta\phi} = -\frac{2\pi e}{k} \delta^{(2)}(x), \quad \bar{F}_{\theta\phi} = \frac{2\pi e}{k} \delta^{(2)}(x). \quad (71)$$

Thus, the holonomies of the gauge fields around a constant θ circle are given by

$$W_A = e^{i \oint A} = e^{i \oint F} = e^{\frac{2i\pi e}{k}}, \quad W_{\bar{A}} = e^{i \oint \bar{F}} = e^{-\frac{2i\pi e}{k}}. \quad (72)$$

In the gauge $A_\theta = 0$ this implies $A_\phi = \frac{e}{k}$ and $\bar{A}_\phi = -\frac{e}{k}$ outside the center of AdS_3 where the gauge field is flat. The time components are fixed by the boundary conditions at the AdS boundary. Thus, the gauge fields due to a classical point particle are given by

$$A = \frac{e}{k}(dt + d\phi), \quad \bar{A} = \frac{e}{k}(dt - d\phi). \quad (73)$$

The gauge field contribution to the holographic energy momentum tensor separates from the rest of the energy momentum tensor since the Chern-Simons action is independent of the spacetime metric. Thus, the Chern-Simons contribution to the

⁶Strictly speaking we should introduce a multiplicative renormalization factor relating $(\mathcal{O}_1 \mathcal{O}_2)(x) = \lim_{\epsilon \rightarrow 0} Z(\epsilon) \mathcal{O}_1(x + \epsilon) \mathcal{O}_2(x)$, to have a finite two point function in the limit $\epsilon \rightarrow 0$.

holographic energy momentum tensor appears from a boundary term, and is given by⁷ [7]

$$T_{\mu\nu} = \frac{k}{8\pi} \left(A_\mu A_\nu + \bar{A}_\mu \bar{A}_\nu - A_\mu \bar{A}_\nu - A_\nu \bar{A}_\mu - \frac{1}{2} \eta_{\mu\nu} (A^2 + \bar{A}^2 - 2A \cdot \bar{A}) \right). \quad (74)$$

This gives us the Chern-Simons contribution to the energy of the particle as

$$E_{CS}(e) = \int_0^{2\pi} T_{tt} = \frac{e^2}{2k}. \quad (75)$$

Two particles with charges e_1 and e_2 give rise to the same holonomy as a single particle with charge $e_1 + e_2$. Thus, the gauge fields near the boundary have the same form (73) with $e = e_1 + e_2$. Due to the non-linearity of the holographic energy momentum tensor of Chern-Simons theory, there is an interaction energy between two charged particles

$$\delta E = E_{CS}(e_1 + e_2) - E_{CS}(e_1) - E_{CS}(e_2) = \frac{e_1 e_2}{k}, \quad (76)$$

This is indeed the anomalous dimension we obtain from the 4-point function.

8 Discussion

The effect of bulk Chern-Simons fields in the scalar four point function is reproduced in the leading order in $1/k$ by the operator product expansion of the operators 1 , and j_μ and its (global) conformal descendants. Since the CFT under consideration must also include the scalar operators \mathcal{O}_1 and \mathcal{O}_2 , we could have very well added operators of the form

$$\mathcal{O}_i^\dagger \overleftrightarrow{\square}^n \overleftrightarrow{\partial}_{\mu_1} \dots \overleftrightarrow{\partial}_{\mu_l} \mathcal{O}_i - \text{traces} \quad (77)$$

to the OPE. Indeed in order to satisfy crossing symmetry of the OPE one must include operators of the form

$$\mathcal{O}_1 \overleftrightarrow{\square}^n \overleftrightarrow{\partial}_{\mu_1} \dots \overleftrightarrow{\partial}_{\mu_l} \mathcal{O}_2 - \text{traces}, \quad (78)$$

in the $|z_{14}| \rightarrow 0$ OPE channel, as discussed in Appendix E. Thus, from the point of view of a generic CFT there is no reason for leaving out operators of the form (77) from the $|z_{12}| \rightarrow 0$ OPE. Including the operators (77) would lead to an infinite number of possible four point functions. These four point functions must satisfy crossing symmetry, which gives an infinite number of constraints still leading to an infinite number of possible four point functions.

⁷We assume below that k is positive

Pure Chern-Simons interaction between the scalars thus corresponds to a particularly minimal operator content in the OPE. Conformal blocks of most CFT operators are not crossing symmetric by themselves⁸ and thus cannot give rise to a consistent four point function without the addition of an infinite set of further operators [11]. Thus, most bulk particle exchange interactions cannot be directly interpreted as due to the conformal block of the corresponding operator dual to the bulk field being exchanged. The key simplification in the case of the current operator in 1+1 dimensions is that its conformal block is crossing symmetric (with an appropriate choice of composite operators (78) in the other $z_{14} \rightarrow 0$ channel). It is straightforward to demonstrate that the current conformal block in 1+1 dimensions is crossing symmetric. The details are included in Appendix E.

As shown in Appendix E one can read off the anomalous dimensions of all the operators of the form (78) from the four point function (65)

$$\gamma(n, l) = \frac{e_1 e_2}{k}. \quad (79)$$

As discussed in [10], the l dependence of the anomalous dimension is related to how the corresponding bulk interaction varies with bulk (proper) distance. This follows as at sufficiently large l the distance between the bulk fields in the state dual to the operator (78) scales as $b \propto \log l$. If we had started from the CFT side we could ask what kind of a bulk interaction could reproduce the contribution of the current conformal block. Knowing the anomalous dimensions (79) from the block, we would be lead to the conclusion that whatever bulk interaction is dual to it, the interaction has to be independent of the bulk distance between the particles.⁹ This is of course consistent with Chern-Simons interaction, which as we saw in section 5, has no local bulk effects but simply shifts the energies of all charged bulk particle states by a constant.

Acknowledgements

I would like to thank A. O' Bannon, A. Starinets and K.P. Yogendran for discussions. This research was supported by the European Research Council under the European

⁸Meaning with any operator content in the other $z_{14} \rightarrow 0$ channel.

⁹Strictly speaking this can be concluded only for large l , but even for smaller l even though no obvious quantitative connection between bulk distance and l exists, it still seems likely that the only way to get anomalous dimensions independent of l and n is to have an interaction independent of distance.

A Some details on conventions

Integral measures d^3x are always defined to be positive. Complex coordinates have been defined through

$$z = x + it_E, \quad w = \tau + i\phi. \quad (80)$$

These may be continued to real time using $t_L = -it_E$ and $t = -i\tau$. Throughout we assume (without loss of generality) that the level k is positive. All the ϵ symbols we use are defined as

$$\epsilon^{uz\bar{z}} = 1, \quad \epsilon^{uxt_L} = 1, \quad \epsilon^{t\theta\phi} = 1. \quad (81)$$

The boundary terms are evaluated at $u = \epsilon$, where the induced metric is $\gamma_{z\bar{z}} = 1/(2\epsilon^2)$, $\gamma^{z\bar{z}} = 2\epsilon^2$ and $\sqrt{|\gamma|} = 1/(2\epsilon^2)$.

B Current OPE coefficient

In this Appendix we review how the $\mathcal{O}^\dagger j \mathcal{O}$ OPE coefficient is fixed by a current conservation Ward identity in the CFT. The OPE coefficient is defined through the 3-point function

$$\langle \mathcal{O}^\dagger(x_1) j^\mu(x_2) \mathcal{O}(x_3) \rangle = C_{\mathcal{O}^\dagger j \mathcal{O}} \frac{Z^\mu}{|x_1 - x_3|^{2\Delta_i}}, \quad Z^\mu = \frac{x_2^\mu - x_3^\mu}{(x_2 - x_3)^2} - \frac{x_2^\mu - x_1^\mu}{(x_2 - x_1)^2}. \quad (82)$$

Current conservation Ward identities tell us that

$$\langle \mathcal{O}^\dagger(x_1) \partial_\mu j^\mu(x_2) \mathcal{O}(x_3) \rangle = -ie(\delta^{(2)}(x_2 - x_3) - \delta^{(2)}(x_2 - x_1)) \langle \mathcal{O}^\dagger(x_1) \mathcal{O}(x_2) \rangle. \quad (83)$$

This Ward identity can be derived for example using standard path integral methods [13] and we have used the fact the \mathcal{O} transforms as $\mathcal{O} \rightarrow e^{ie\alpha} \mathcal{O}$ under the global symmetry transformations. Requiring (82) to be consistent with (83) fixes the OPE coefficient $C_{\mathcal{O}^\dagger j \mathcal{O}}$ since

$$\partial_{\mu, x_2} \langle \mathcal{O}^\dagger(x_1) j^\mu(x_2) \mathcal{O}(x_3) \rangle = -2\pi C_{\mathcal{O}^\dagger j \mathcal{O}} \frac{1}{|x_1 - x_2|^{2\Delta_i}} (\delta^{(2)}(x_2 - x_3) - \delta^{(2)}(x_2 - x_1)), \quad (84)$$

where we used $\partial_{\mu,x}(x^\mu - y^\mu)/(x - y)^2 = -2\pi\delta^{(2)}(x - y)$. Comparing to the Ward identity gives

$$C_{\mathcal{O}^\dagger j \mathcal{O}} = i \frac{e}{2\pi}. \quad (85)$$

The OPE coefficient is symmetric under the change of its indices, except when changing the order of \mathcal{O}^\dagger and \mathcal{O} , under which the OPE coefficient changes sign

$$C_{\mathcal{O} j \mathcal{O}^\dagger} = -C_{\mathcal{O}^\dagger j \mathcal{O}} = -i \frac{e}{2\pi} \quad (86)$$

C OPE coefficients between the scalars and the current operator

The operator product expansion of a charged scalar operator \mathcal{O} in a generic 1+1 CFT includes the contributions of the current operator

$$\mathcal{O}^\dagger(z_1)\mathcal{O}(z_3) = \frac{1}{|z_{13}|^{2\Delta}} \sum_n c_n z_{13}^{n+1} \partial^n j_z(z_3) + \dots, \quad (87)$$

where the dots correspond to other operators in the theory and include the same sum for the right moving component of the current $j_{\bar{z}}$. Since the analysis for the right moving part is identical to the one for the left moving component, we will only discuss the latter here in detail. Also (87) only includes the global conformal descendants $L_{-1}^k j_z$. The reason for leaving out the higher Virasoro descendants L_{-n}^k is that they give rise to subleading contributions in the limit of large central charge. This is explained in more detail in the Appendix D.

To fix the unknown coefficients c_n , we compare the full three point function

$$\langle \mathcal{O}^\dagger(z_1) j_z(z_2) \mathcal{O}(z_3) \rangle = \frac{C_{\mathcal{O}^\dagger j \mathcal{O}}}{2} \frac{1}{|z_{13}|^{2\Delta}} \left(\frac{1}{z_{23}} - \frac{1}{z_{21}} \right), \quad (88)$$

to the one obtained from operator product expansion

$$\langle \mathcal{O}^\dagger(z_1) j_z(z_2) \mathcal{O}(z_3) \rangle = \frac{1}{|z_{13}|^{2\Delta}} \sum_n c_n z_{13}^{n+1} \langle \partial^n j_z(z_3) j_z(z_2) \rangle. \quad (89)$$

Using the known current two point function gives

$$\langle \mathcal{O}^\dagger(z_1) j_z(z_2) \mathcal{O}(z_3) \rangle = -\frac{C_V}{2} \frac{1}{|z_{13}|^{2\Delta}} \sum_n c_n (n+1)! z_{13}^{n+1} z_{23}^{-n-2}. \quad (90)$$

On the other hand series expanding (88) gives

$$\langle \mathcal{O}^\dagger(z_1) j_z(z_2) \mathcal{O}(z_3) \rangle = -\frac{C_{\mathcal{O}^\dagger j \mathcal{O}}}{2} \frac{1}{|z_{13}|^{2\Delta}} \sum_n z_{13}^{n+1} z_{23}^{-n-2}. \quad (91)$$

Matching the coefficients of the two series fixes the OPE coefficients

$$c_n = \frac{C_{\mathcal{O}^\dagger j \mathcal{O}}}{(n+1)! C_V}. \quad (92)$$

D Decoupling of Virasoro descendants at large c

This can be straightforwardly seen from the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{2}n(n^2-1)\delta_{n+m,0}. \quad (93)$$

As an example consider the descendant $L_{-n}j_z$. The two point function of this operator is proportional to the norm

$$\langle L_{-n}j_z | L_{-n}j_z \rangle = \langle j_z | L_n L_{-n} | j_z \rangle = \left(\frac{c}{2}n(n^2-1) + 2n\right) \langle j_z | j_z \rangle. \quad (94)$$

Thus, for $n > 1$, the two point function scales as c^1 as $c \rightarrow \infty$.

On the other hand the three point functions

$$\langle \mathcal{O}^\dagger(z_1) L_{-n}j_z(z_2) \mathcal{O}(z_3) \rangle \quad (95)$$

are obtained from (88), by acting with derivative operators [13]. Thus, as far as counting powers of c goes, these three point functions are order c^0 . Using the same procedure as in the previous appendix to obtain the OPE coefficients of the higher descendants by considering (95), we find that the OPE coefficients are of the order c^{-1} . Thus, we can ignore them in the leading order at large c .

E On crossing symmetry

We have not yet shown whether the operator content we took into account is consistent, in the sense that it should satisfy crossing symmetry. This is the purpose of this Appendix.

Using the decomposition of the 4 point function in (45), the OPE can be organized

into primary operators

$$g(u, v) = \sum_{\mathcal{O}} C_{11\mathcal{O}} C_{22\mathcal{O}} g_{\Delta, l}(\eta, \bar{\eta}), \quad (96)$$

where Δ and l are the scaling dimension and the spin of the primary operator and

$$u = \eta\bar{\eta}, \quad v = (1 - \eta)(1 - \bar{\eta}). \quad (97)$$

The function $g_{\Delta, l}(\eta, \bar{\eta})$ is given by [16]

$$g_{\Delta, l}(\eta, \bar{\eta}) = \left(-\frac{1}{2}\right)^l 2^{-\delta_{l0}} \left(k(\Delta + l, \eta)k(\Delta - l, \bar{\eta}) + k(\Delta + l, \bar{\eta})k(\Delta - l, \eta)\right), \quad (98)$$

where we used the notation [9]

$$k(\beta, \eta) = \eta^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta, \eta). \quad (99)$$

This is the operator product expansion in the S-channel (see Figure 2). We can also perform the OPE in the region $|z_{14}| \rightarrow 0$ and $|z_{23}| \rightarrow 0$. This is called the T-channel OPE and is given in terms of the conformal blocks as

$$g(u, v) = \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{1^\dagger 2^\dagger \mathcal{O}^\dagger} f_{\Delta, l}(\eta, \bar{\eta}), \quad (100)$$

where

$$f_{\Delta, l}(\eta, \bar{\eta}) = (-1)^l (\eta\bar{\eta})^{\Delta_2} ((1 - \eta)(1 - \bar{\eta}))^{\Delta - \Delta_1 - \Delta_2 - l} F(\Delta, l), \quad (101)$$

and

$$F(\Delta, l) = \frac{1}{2^l 2^{-\delta_{l0}}} \left((1 - \eta)^l {}_2F_1(a, a, \Delta + l, 1 - \eta) {}_2F_1(a - l, a - l, \Delta - l, 1 - \bar{\eta}) + (\eta \leftrightarrow \bar{\eta}) \right),$$

where $a = (\Delta + \Delta_2 - \Delta_1 + l)/2$.

Crossing symmetry is the requirement that the four point function should be independent of the channel one chooses to expand on. As an equation, this reads

$$\sum_{\mathcal{O}} C_{11^\dagger \mathcal{O}} C_{22^\dagger \mathcal{O}} g_{\Delta, l}(\eta, \bar{\eta}) = \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{1^\dagger 2^\dagger \mathcal{O}^\dagger} f_{\Delta, l}(\eta, \bar{\eta}). \quad (102)$$

With the operator content $(1, J_\mu)$ the S-channel OPE is simply given by

$$g(u, v) = 1 + \frac{1}{2} \frac{e_1 e_2}{k} \log v. \quad (103)$$

The T-channel OPE involves the primary operators of the schematic form $\mathcal{O}_1(\partial^2)^n \partial_1 \dots \partial_l \mathcal{O}_2$, which have scaling dimensions

$$\Delta(n, l) = \Delta_1 + \Delta_2 + 2n + l + \gamma(n, l) = \Delta_0(n, l) + \gamma(n, l), \quad (104)$$

where $\gamma(n, l)$ are the corresponding anomalous dimensions assumed to be of the order $1/k$. Furthermore we will denote the OPE coefficients as $c(n, l) = (-1)^l C_{12\mathcal{O}} C_{1^\dagger 2^\dagger \mathcal{O}^\dagger} = |C_{12\mathcal{O}}|^2$. Defined this way, the coefficients $c(n, l)$ must be positive. Thus, the crossing symmetry equation takes the form of a double sum

$$\sum_{n, l} c(n, l) u^{\Delta_2} v^{n + \frac{1}{2}\gamma(n, l)} F(\Delta_0(n, l) + \gamma(n, l), l) = 1 + \frac{1}{2} \frac{e_1 e_2}{k} \log v. \quad (105)$$

We would like to solve this equation order by order in $1/k$. Thus, we expand the OPE coefficients as $c(n, l) = c_0(n, l) + \delta c(n, l)$, where $\delta c(n, l)$ is of the order $1/k$.

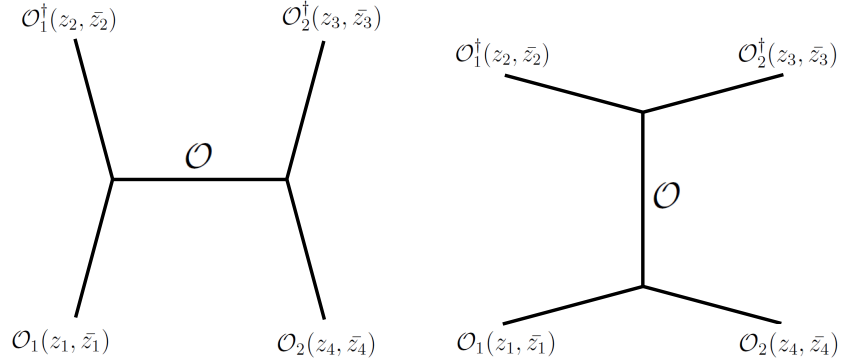


Figure 2: Left: OPE in the S-channel. Right: OPE in the T-channel. The OPE sums over the possible operators \mathcal{O} .

Zeroth order: To zeroth order in the $1/k$ expansion we get

$$\sum_{n, l} c_0(n, l) v^n F(\Delta_0, l) = u^{-\Delta_2} \quad (106)$$

This equation can be uniquely solved for the OPE coefficients $c_0(n, l)$. Let us define the variables $a = 1 - \eta$ and $b = 1 - \bar{\eta}$. We will solve (106) by series expanding around $a = 0$ and $b = 0$.

Using the hypergeometric series¹⁰ ${}_2F_1(a, b, c, x) = \sum_{n=0}^{\infty} (a)_n (b)_n / (c)_n x^n / n!$ it is

¹⁰We use the following convention for the Pochhammer symbol $(x)_n = \Gamma(x + n) / \Gamma(x)$

straightforward to expand (106) as a power series in a and b

$$\sum_{m,k} h_1(m,k) a^m b^k = \sum_{n,l,k,m} c_0(n,l) h_2(n,l,m,k) \left(a^{k+n} b^{m+l+n} + a^{m+l+n} b^{k+n} \right), \quad (107)$$

where all of the sums range from 0 to ∞ , and

$$h_1(m,k) = \frac{\Gamma(\Delta_2 + m) \Gamma(\Delta_2 + k)}{\Gamma(\Delta_2)^2 m! k!}, \quad h_2(n,l,m,k) = \frac{(\alpha)_k^2 (\beta)_m^2}{2^{l+\delta_{l0}} (\beta)_k (\delta)_m k! m!}. \quad (108)$$

where $\alpha = \Delta_2 + n + l$, $\beta = \Delta_1 + \Delta_2 + 2n + 2l$, $\gamma = \Delta_2 + n$ and $\delta = \Delta_1 + \Delta_2 + 2n$. The equality in (107) should hold for each power of a and b separately. This leads us to the following method of solving (107) iteratively. Consider first the powers $a^0 b^j$ in (107), for some fixed j , which gives

$$h_1(0,j) = \sum_{l=0}^j h_2(0,l,j-l,0) c_0(0,l) + h_2(0,0,0,j) c_0(0,0). \quad (109)$$

This can be solved iteratively to give $c_0(0,l)$ for all values of l . Taking $j = 0$ gives $h_1(0,0) = 2h_2(0,0,0,0)c_0(0,0)$, which after using (108) gives $c_0(0,0) = 1$. Next we can solve $c_0(0,1)$, $c_0(0,2)$ and so forth, up to arbitrarily high order. Thus, we can in principle determine all $c_0(0,j)$ from (109). For example the first few terms are

$$c_0(0,0) = 1, \quad c_0(0,1) = \frac{2\Delta_1\Delta_2}{\Delta_1 + \Delta_2}, \quad c_0(0,2) = \frac{2\Delta_1(\Delta_1 + 1)\Delta_2(\Delta_2 + 2)}{(1 + \Delta_1 + \Delta_2)(2 + \Delta_1 + \Delta_2)}. \quad (110)$$

Now we can consider the $a^i b^j$ term in (107). The key is to note that on the right hand side the sum over n gives non-vanishing contributions only when $n = 0, \dots, i$. This means that only the coefficients $c(n,l)$ with $n \leq i$ and $l \leq j$ appear in (107). Thus, we can again first take $i = 1$ and solve the equations for all j and then move to $i = 2$ and solve for all j and so on. Thus, in principle, all of the coefficients $c_0(n,l)$ are uniquely determined by crossing symmetry. As so far we are considering a generalized free field theory, we will not pursue the specific forms of $c_0(n,l)$, but will simply assume that the unique solution for the coefficients is consistent, i.e. $c_0(n,l) \geq 0$.

First order: To first order in $1/k$ the crossing symmetry equation (105) becomes

$$\sum_{n,l} \left(\delta c(n,l) v^n F(\Delta_0, l) + \frac{1}{2} c_0(n,l) \gamma(n,l) v^n F(\Delta_0, l) \log v + \frac{1}{2} c_0(n,l) \gamma(n,l) v^n \frac{\partial}{\partial n} F(\Delta_0, l) \right) = \frac{1}{2} \frac{e_1 e_2}{k} \log v, \quad (111)$$

Identifying the logarithmic parts¹¹ gives

$$\sum_{n,l} v^n c_0(n,l) \gamma(n,l) F(\Delta_0, l) = u^{-\Delta_2} \frac{e_1 e_2}{k}. \quad (112)$$

Comparing this to (106) immediately gives us the unique solution for the anomalous dimensions

$$\gamma(n,l) = \frac{e_1 e_2}{k}. \quad (113)$$

Thus, all of the composite higher spin operators have the same anomalous dimension. Finally we can solve for the $\delta c(n,l)$ coefficients from the rest of (111)

$$\sum_{n,l} v^n \left(\delta c(n,l) F(\Delta_0, l) + \frac{1}{2} c_0(n,l) \gamma(n,l) \frac{\partial}{\partial n} F(\Delta_0, l) \right) = 0. \quad (114)$$

Using again $a = 1 - \eta$ and $b = 1 - \bar{\eta}$ and series expanding gives

$$\sum_{n,l,k,m} \left[\left(\delta c(n,l) h_2(n,l,m,k) + \frac{1}{2} \gamma(n,l) c(n,l) \partial_n h_2(n,l,m,k) \right) \times \right. \\ \left. \times (a^{k+n} b^{m+l+n} + a^{m+l+n} b^{k+n}) \right] = 0, \quad (115)$$

We can use the same iterative procedure to uniquely solve the equation for $\delta c(n,l)$ as we used to solve for $c_0(n,l)$ as the equations have the same structure in terms of powers of a and b . The few lowest coefficients are given by

$$\delta c(0,0) = 0, \quad \delta c(0,1) = -\frac{2e_1 e_2 \Delta_1 \Delta_2}{k(\Delta_1 + \Delta_2)^2}, \\ \delta c(0,2) = -\frac{2e_1 e_2 \Delta_1 (\Delta_1 + 1) \Delta_2 (\Delta_2 + 1) (3 + 2\Delta_2 + 2\Delta_1)}{k(\Delta_1 + \Delta_2 + 1)^2 (\Delta_1 + \Delta_2 + 2)^2}. \quad (116)$$

This procedure is guaranteed to give a solution to all of the coefficients $\delta c(n,l)$ because the coefficients $h_2(n,l,m,k)$ in front of $\delta c(n,l)$ are all non-vanishing. Thus, we have demonstrated (up to the caveat that we have not shown that $c_0(n,l)$ are positive) that

¹¹This can be done because the $F(\Delta_0, l)$ does not have logarithmic terms near $v = 0$.

the four point function with only the operators $(1, j_\mu)$ in the S channel OPE, satisfies crossing symmetry with unique values for the OPE coefficients and the anomalous dimensions (113).

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